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CRITICAL RIEMANNIAN MANIFOLDS

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0. Preliminary motivation

The idea for this paper grew out of the conviction that solutions of naturally posed variational problems ought to be nice in some respect, such as symmetry. Consider the intrinsic variational problem of finding those metrics g on a compact n-manifold M (with the volume of (M, g) prescribed) for which the function $I(g) = \int_M R$ ($R =$ scalar curvature of g) is stationary. The well-known necessary and sufficient condition is that (M, g) is an Einstein space ($R_{ij} = Rg_{ij}/n$). It is natural to ask whether there are any Riemannian manifolds which are so nice that they will be solutions of *every* reasonable intrinsic variational problem. The precise meaning of "reasonable" is given in §1. Certainly the function which defines the problem should assign the same value to isometric Riemannian manifolds, and a certain smoothness of the function is required. At any rate, with the definition of "reasonable" which is adopted, we prove that (M, g) is a solution of every reasonable intrinsic variational problem (i.e., (M, g) is a critical Riemannian manifold) iff (M, g) is a homogeneous space whose isotropy group at a point acts irreducibly on the tangent space of M at that point (i.e., (M, g) is a isotropy irreducible homogeneous space–see [8] and [9]). A rough outline of the proof is contained in §1. At this point it suffices to say that, aside from essential and repeated use of the Ebin-Palais slice theorem [3] for the action of diffeomorphisms on metrics via pull-back, the rest of the proof lies quite near the surface of what is commonly known.

1. Notational background and a precise statement

Let M be a fixed compact oriented C^∞ n-manifold, and let $\mathfrak{M}$ be the manifold of C^∞ Riemannian metrics on M. Set $\mathfrak{D} \equiv$ Lie group of C^∞ diffeomorphisms of M. The sense in which $\mathfrak{M}$ is a manifold and $\mathfrak{D}$ is a Lie group is discussed in [2], [3], and [6]. Observe that $\mathfrak{M}$ is an open subset of $\Gamma(S^2M) \equiv$ vector space of C^∞ symmetric covariant two-tensors. Thus the

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tangent space of $\mathfrak{M}$ at any $g \in \mathfrak{M}$ may be identified with $\Gamma(S^2M)$. A function $f\colon \mathfrak{M} \to \mathbf{R}$ is said to be *smooth* if at each $g \in \mathfrak{M}$ there is a tangent vector $(\nabla f)_g \in \Gamma(S^2M)$ such that

$$\lim_{t\to 0} \frac{1}{t}(f(g+ts) - f(g)) = \langle (\nabla f)_g, s\rangle_g$$

for all $s \in \Gamma(S^2M)$, where $\langle h, k\rangle_g \equiv \int_M h_{ij}k_{pm}g^{pi}g^{mj}\omega_g$ for $h, k \in \Gamma(S^2M)$, ω_g being the volume element of (M, g). Note that $g = \langle\, ,\, \rangle_g$ is a Riemannian metric on $\mathfrak{M}$ in the sense of [3]. Also, $\mathfrak{D}$ acts on $\mathfrak{M}$ to the right via pull-back: $g \cdot \eta = \eta^*(g)$. A calculation [3, p. 20] shows that $g \to \eta^*(g)$ is an isometry of $\mathfrak{M}$ for each $\eta \in \mathfrak{D}$. A smooth $f\colon \mathfrak{M} \to \mathbf{R}$ is said to be *invariant* if $f(\eta^*g) = f(g)$ for all $\eta \in \mathfrak{D}$ and for all $g \in \mathfrak{M}$. If, in addition, $f(\lambda g) = f(g)$ for all $\lambda \in \mathbf{R}^+$ and all $g \in \mathfrak{M}$, then f is said to be *fully-invariant*. The Riemannian manifold (M, g) is said to be *critical* if $(\nabla f)_g = 0$ for every smooth fully-invariant $f\colon \mathfrak{M} \to \mathbf{R}$. If we had omitted the prefix "fully" in this definition, no Riemannian manifold would be critical, since the smooth invariant function $V\colon \mathfrak{M} \to \mathbf{R}$ with $V(g) \equiv \text{Volume } (M, g)$ has $(\nabla V)_g = \frac{1}{2}g \neq 0$ by a calculation in §3.

Recall that an isotropy-irreducible homogeneous space (IHS) is a homogeneous space whose isotropy group at a point acts irreducibly on the tangent space at that point. For a full account of this rich class of spaces see [8]. The main result of this paper is

Theorem 1.1. *(M, g) is critical $\Leftrightarrow$ (M, g) is an IHS.*

The implication "$\Leftarrow$" is proved in §2. The authors's first approach in trying to prove "$\Rightarrow$" was to find a large number of fully-invariant functions formed by integrating various scalars built from the curvature tensor and normalizing them by dividing by appropriate powers of the volume. Then one computes the gradients and sets them equal to $0 \in \Gamma(S^2M)$, obtaining a list of conditions which might imply (M, g) is locally an IHS. Whether this approach will eventually work remains to be seen. Some of the more interesting results of this effort are summarized in §3. §4 contains a proof of "$\Rightarrow$" in Theorem 1.1 and may be read independently of §3. The proof is based on the Ebin-Palais slice theorem(s) for the action of $\mathfrak{D}$ on $\mathfrak{M}$ (see Theorem 7.4 and remark 2 on [3, p. 34]).

The idea of the proof of "$\Rightarrow$" in Theorem 1.1 is briefly explained in this paragraph. If (M, g) is not homogeneous, then using the above mentioned slice theorem and the theorem that the group of conformal transformations of a compact Riemannian manifold is a *finite* dimensional Lie group, we can prove the existence of a section $s \in \Gamma(S^2(M))$ with $\operatorname{div} s = 0$, $s \neq 0$, $\langle s, g\rangle_g = 0$, and $\eta^*s = s$ for all isometries η of (M, g). Using the slice theorem again we can construct a smooth fully-invariant function $f\colon \mathfrak{M} \to \mathbf{R}$ with $(\nabla f)_g = s$ whence (M, g) is not critical. If (M, g) is homogeneous but not an *IHS*, we prove the existence of such an s by an interesting algebraic argument, and construct f as above.

2. IHS $\Rightarrow$ critical

A Riemannian manifold (M, g) is said to be *solo* if the only sections $s \in \Gamma(S^2M)$ with $\eta^*s = s$ for all $\eta \in I_g$ ($\equiv$ group of isometries of (M, g)) are the real multiples of g. We begin with

Lemma 2.1. *(M, g) is solo $\Leftrightarrow$ (M, g) is an* IHS.

Proof. If (M, g) is not homogeneous, then it is an easy matter to construct a positive nonconstant function $F\colon M \to \mathbf{R}$ invariant under I_g, whence Fg is invariant under I_g, and so (M, g) is not solo. Thus solo $\Rightarrow$ homogeneous, and therefore we need only prove the Lemma for (M, g) homogeneous. Thus, let (M, g) be homogeneous, but not an IHS. Then the isotropy subgroup $K \subset I_g$ at some point $p \in M$ leaves a proper nonzero subspace of T_pM fixed. Translating this subspace to all points of M via isometries yields a well-defined distribution $\{V_q\}$ of subspaces because of the K-invariance at p. Define $s_q \in S_q^2M$ by $s_q(X, Y) = g(\pi X, \pi Y)$ where $\pi\colon T_qM \to V_q$ is orthogonal projection. Then it is easy to prove $\eta^*s = s$ for all $\eta \in I_g$, but $s \neq \lambda g$. Thus (M, g) is not solo, and hence we have solo $\Rightarrow$ IHS. The three-line proof that IHS $\Rightarrow$ solo is found on [8, p. 137].

Lemma 2.2. *If (M, g) is solo, then (M, g) is critical.*

Proof. Let f be a smooth fully-invariant function on $\mathfrak{M}$. Recall $(\nabla f)_g \in \Gamma(S^2M)$. We first prove $(\nabla f)_g$ is invariant under I_g. Let $\eta \in I_g$. Since f is invariant under $\mathfrak{D}$, in particular $f \circ \eta^*\colon \mathfrak{M} \to \mathbf{R}$ is $f\colon \mathfrak{M} \to \mathbf{R}$. Thus for all $s \in \Gamma(S^2M)$ we have

$$\begin{aligned}\langle(\nabla f)_g, s\rangle_g &= \langle(\nabla(f \circ \eta^*))_g, s\rangle_g \\ &\equiv \lim_{t\to 0} \frac{1}{t}\big((f \circ \eta^*)(g + ts) - (f \circ \eta^*)(g)\big) \\ &= \lim_{t\to 0} \frac{1}{t}\big(f(\eta^*g + t\eta^*s) - f(\eta^*g)\big) \\ &= \lim_{t\to 0} \frac{1}{t}\big(f(g + t\eta^*s) - f(g)\big) \\ &= \langle(\nabla f)_g, \eta^*s\rangle_g = \langle(\eta^{-1})^*(\nabla f)_g, s\rangle_g,\end{aligned}$$

where the last equaltiy follows from the definition of $\langle\ ,\ \rangle_g$ and the fact that $\eta \in I_g$. From the above calculation, it follows easily that $(\nabla f)_g = (\eta^{-1})^*(\nabla f)_g$

for all $\eta \in I_g$. Thus (M, g) solo implies $(\nabla f)_g = \lambda g$. However, using the *full* invariance of f, we have

$$nV(g)\lambda = \langle \lambda g, g\rangle_g = \langle (\nabla f)_g, g\rangle_g = \lim_{t\to 0} \frac{1}{t}(f(g + tg) - f(g)) = 0.$$

Thus $\lambda = 0$, and so $(\nabla f)_g = 0$ for all smooth fully-invariant functions f.

Theorem 2.3. *If* (M, g) *is an* IHS, *then* M *is critical.*

Proof. This is clear from the lemmas.

It is easy to strengthen Lemma 2.2. In the definition of solo, if we add the additional requirement that $\operatorname{div} s = 0 \in \Gamma(T^*M)$, we say that (M, g) is *semi-solo*. The stronger version of Lemma 2.2 is

Theorem 2.4. *If* (M, g) *is semi-solo, then* (M, g) *is critical.*

Proof. This follows immediately from the next lemma and the proof of Lemma 2.2.

Lemma 2.5. *If* f *is a smooth invariant function on* $\mathfrak{M}$, *then* $\operatorname{div}(\nabla f)_g = 0$ *where* $\operatorname{div}(\nabla f)_g$ *is the divergence, computed with respect to the metric* g, *of* $(\nabla f)_g \in \Gamma(S^2M)$.

Proof. It is not difficult to prove the formula

$$\langle s, L_X g\rangle_g = -2\int_M (\operatorname{div} s)(X)\omega_g$$

where X is a vector field and L_X is Lie differentiation. This formula is equivalent to the formula for the adjoint of $\delta \equiv -\operatorname{div}\colon \Gamma(S^2M) \to \Gamma(T^*M)$ given on [2, p. 380].

Let η_t be the one-parameter group of diffeomorphisms of M generated by X. The invariance of f yields

$$0 = \frac{d}{dt} f(\eta_t^* g)\Big|_{t=0} = \langle (\nabla f)_g, L_X g\rangle_g = -2\int_M (\operatorname{div}(\nabla f)_g)(X)\omega_g$$

for all vector fields X. Thus $\operatorname{div}(\nabla f)_g = 0$. q.e.d.

The converse of Theorem 2.4 is proved in §4. The reader may proceed to §4 at this point, or read through the examples of §3.

3. Examples of functions on $\mathfrak{M}$

Here we give a number of examples of invariant and fully-invariant functions f on $\mathfrak{M}$, and the results of the somewhat tedious computations of the gradients $(\nabla f)_g$.

(A) Let $V\colon \mathfrak{M} \to \mathbf{R}$ be given by $V(g) = \int_M \omega_g = \operatorname{vol}(M, g)$. Observe that for matrices A and B, $\det(A + tB) = \det(A)\det(I + tA^{-1}B)$. Then

$$\frac{d}{dt}\det(A + tB)_{t=0} = \det(A)\operatorname{trace}(A^{-1}B).$$

Let $A = (g_{ij})$ and $B = (s_{ij})$, where $g_{ij} = g(\partial_i, \partial_j)$ for coordinate vector fields ∂_i, and $s_{ij} = s(\partial_i, \partial_j)$ for $s \in \Gamma(S^2M)$. Then the above formula yields

$$\begin{aligned}\frac{d}{dt}\omega_{g+ts}\bigg|_{t=0} &= \frac{d}{dt}\left[\det(A + tB)\right]^{1/2}\bigg|_{t=0} dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{1}{2}(\det A)^{-1/2}\det A \operatorname{trace}(A^{-1}B)\, dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{1}{2}\left[\det(g_{ij})\right]^{1/2} g^{ij}s_{ij}\, dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{1}{2} g^{ij}s_{ij}\omega_g.\end{aligned}$$

Thus

$$\frac{d}{dt}V(g + ts)\bigg|_{t=0} = \int_M \frac{1}{2} g^{ij}s_{ij}\omega_g = \left\langle \frac{1}{2}g, s\right\rangle_g,$$

and so $(\nabla V)_g = \frac{1}{2}g$.

(B) Let $f(g) = \int_M R\omega_g$ where $R = R^{ij}{}_{ij}$ is the scalar curvature of (M, g). A well-known fact (see [5]) is that $(\nabla f)_g = -(R_{ij} - \frac{1}{2}Rg_{ij})$ where $R_{ij} = R^k{}_{ikj}$ is the Ricci curvature. The function $f_0(g) = V(g)^{2/n-1}\int_M R\omega_g$ is fully-invariant, and using the product rule for differentiation we find that

$$V(g)^{1-2/n}(\nabla f_0)_g = -R_{ij} + \left[\frac{1}{n}\bar{R} + \frac{1}{2}(R - \bar{R})\right] g_{ij},$$

where $\bar{R} = V(g)^{-1}\int_M R\omega_g$. Thus we see that (M, g) is critical for f_0 iff (M, g) is Einstein. This result is well-known, and Mutō [5] has even studied the second variation.

(C) Let $f(g) = V(g)^{4/n-1}\int_M R_{ij}R^{ij}\omega_g$. We can compute the gradient of this fully-invariant function by using the formula (which appears in [7]) derived by J. C. Du Plessis for the gradient of the unnormalized function $g \to \int_M R_{ij}R^{ij}\omega_g$. If (M, g) is Einstein, it is easy (using this formula) to verify that $(\nabla f)_g = 0$ is automatic. So this function provides no new condition on the curvature tensor of a critical Riemannian manifold.

(D) To his knowledge, no one but the author has bothered to compute the gradient of the fully-invariant function $f(g) = V(g)^{4/n-1}\int_M \|R\|^2\omega_g$ where $\|R\|^2 = R_{ijkm}R^{ijkm}$. The formulas of H. Rund [7], for computing gradients of integrals of densities involving two or fewer derivatives of the g_{ij}, were useful in obtaining the result:

$$\begin{aligned}V(g)^{4/n-1}(\nabla f)_g = {}& 4R_i{}^{hk}{}_j|_{hk} - 2R_i{}^{hkm}R_{jhkm} \\ &+ \left[\tfrac{2}{n}\overline{\|R\|^2} + \tfrac{1}{2}\left(\|R\|^2 - \overline{\|R\|^2}\right)\right] g_{ij},\end{aligned}$$

where $\overline{\|R\|^2} = V(g)^{-1} \int_M \|R\|^2 \omega_g$, and the vertical bar represents covariant differentiation. If (M, g) is Einstein and $\dim M \geqslant 3$, the second Bianchi identity along with $R_{ij} = \lambda g_{ij}$ (λ = const.) can be used to show $R_i{}^{hk}{}_j|_h = 0$, whence the first term of $(\nabla f)_g$ vanishes. Thus, when $n \neq 4$, setting $(\nabla f)_g = 0$ and taking the trace of $(\nabla f)_g$ yields $\|R\|^2 = \overline{\|R\|^2}$, i.e., the curvature tensor has constant length. Moreover, we may then set $\|R\|^2 = \overline{\|R\|^2}$ in the expression for $(\nabla f)_g$ to obtain the stronger condition $R_i{}^{hkm}R_{jhkm} = 1/n\|R\|^2 g_{ij}$ for (M, g) Einstein, $\dim M \geqslant 3$, and critical for f. If $\dim M = 4$, by a tedious calculation, it is found that $R_i{}^{hkm}R_{jhkm} = \frac{1}{n}\|R\|^2 g_{ij}$ follows automatically if (M, g) is Einstein. If $\dim M = 2$, (M, g) is automatically Einstein with $R_{ij} = Kg_{ij}$, K = Gaussian curvature. In this case, M being critical for the function of B) is a restatement of the Gauss-Bonnet Theorem. For the present function, $(\nabla f)_g = 0$ yields the condition $2K|_{ij} = (2\Delta K + K^2 - \overline{K^2})g_{ij}$, where Δ is the Laplacian of g, and $\overline{K^2} = V(g)^{-1} \int_M K^2 \omega_g$. Taking the trace of both sides yields $\Delta K = \overline{K^2} - K^2$, and thus the original equation becomes $2K|_{ij} = (\overline{K^2} - K^2)g_{ij}$. Surprisingly, there are *open* surfaces with nonconstant curvature satisfying this apparently overdetermined equation for *any* constant $\overline{K^2}$, but the author has a proof that the only compact surfaces satisfying the equation have K = const.

(E) D. Lovelock in [4] considers the scalar

$$\delta^{h_1 \cdots h_{2p}}_{j_1 \cdots j_{2p}} R_{h_1 h_2}{}^{j_1 j_2} \cdots R_{h_{2p-1} h_{2p}}{}^{j_{2p-1} j_{2p}},$$

where we have used the generalized Kronecker delta. For brevity we call this scalar H_{2p}. It is essentially the $2p$-th mean curvature of a hypersurface in $\mathbf{R}^{n+1}$. Recall that the $2p$-th mean curvature is intrinsic. When $\dim M = 2p$, H_{2p} is the Gauss-Bonnet integrand up to a constant factor. Lovelock computes the gradient of the invariant function $f(g) = \int_M H_{2p}\omega_g$. The result is simply

$$(\nabla f)_g = \tfrac{1}{2} g_{ik} \delta^{kh_1 \cdots h_{2p}}_{jj_1 \cdots j_{2p}} R_{h_1 h_2}{}^{j_1 j_2} \cdots R_{h_{2p-1} h_{2p}}{}^{j_{2p-1} j_{2p}}.$$

We use this formula to compute the gradiant of the fully-invariant function $f_0 \colon \mathfrak{M} \to \mathbf{R}$, $f_0(g) = V(g)^{2p/n-1} f(g)$. We find easily that

$$V^{1-2p/n}(g)(\nabla f_0)_g = \left(\frac{p}{n} - \frac{1}{2}\right)\overline{H}_{2p} g_{ij} + (\nabla f)_g,$$

where $\overline{H}_{2p} = V(g)^{-1} \int_M H_{2p}\omega_g$. Setting $(\nabla f_0)_g = 0$ and computing the trace of both sides, we obtain $0 = (p - n/2)(\overline{H}_{2p} - H_{2p})$. Hence $H_{2p} = \overline{H}_{2p}$ = const. for $2p \neq n$. If $n = 2p$, then $f = f_0$ and $(\nabla f)_g = 0$ is automatic since the generalized δ will be zero when $n = 2p$. For $n = 2p$, $(\nabla f)_g = 0$ is a restatement of the higher dimensional Gauss-Bonnet Theorem. Resubstituting $H_{2p} = \overline{H}_{2p}$ into the equation $(\nabla f_0)_g = 0$ leads to some simplification in the case $n \neq 2p$, just as in (B) which is the special case $p = 1$.

We summarize the information contained in these examples.

Theorem 3.1. *In order that (M, g) be critical for fully-invariant functions defined by normalized integrals of scalars built from the curvature tensor, it is necessary that*

(1) $R_{ihkm}R_j^{\ hkm} = \lambda g_{ij}$, $n\lambda = \|R\|^2 = \overline{\|R\|^2} =$ *const. for* $n \neq 4$

(2) $\frac{1}{2} g_{ik} \delta_{jj_1 \cdots j_{2p}}^{kh_1 \cdots h_{2p}} R_{h_1 h_2}^{\ \ \ \ j_1 j_2} \cdots R_{h_{2p-1} h_{2p}}^{\ \ \ \ \ \ \ \ j_{2p-1} j_{2p}} = (1/2 - p/n) \overline{H}_{2p} g_{ij}$.

In particular, (M, g) must be Einstein, its curvature tensor must have constant length, and all the mean curvatures H_{2p} of even orders are constant ($n \neq 2p$).

It seems doubtful that these conditions imply that (M, g) is even locally homogeneous, except if $n \leqslant 3$ when they do imply that M has constant curvature. We cannot hope to arrive at the condition that the curvature tensor is parallel, for there are IHS's which are not locally symmetric. One can examine integrals of other scalars in an attempt to meet the local condition for homogeneity which was found by Ambrose and Singer [1]. This line of attack seems worth pursuing because one gets the impression from reading §4 that Theorem 1.1 is made possible because there are too many fully-invariant functions, most of them perhaps having little to do with the usual objects that one studies in geometry (i.e., curvature). Thus it may be more natural to restrict ouselves to fully-invariant functions defined in terms of the curvature, and determine what critical implies in that case.

4. Critical $\Rightarrow$ IHS

In this section there will be many references to the following theorem of Ebin and Palais.

Theorem 4.1 (*Slice theorem*). *Let O_g be the orbit of $g \in \mathfrak{M}$ under $\mathfrak{D}$. Let $\pi: \mathfrak{D} \to O_g$ be given by $\pi(\eta) = \eta^*(g)$. There is a neighborhood U of g in O_g, a section $\chi: U \to \mathfrak{D}$ ($\pi \circ \chi = \mathrm{Id}$), and a neighborhood V of O in the space $H = \{h \in \Gamma(S^2M)\colon \operatorname{div} h = 0\}$, such that (setting $S = g + V$) $F\colon U \times S \to \mathfrak{M}$, defined by $F(u, s) = \chi(u)^*s$, is a diffeomorphism of $U \times S$ onto a neighborhood of g in $\mathfrak{M}$. Moreover, we may assume S has the following properties*:

(1) *For $\eta \in I_g$, $\eta^*S = S$.*

(2) *If $\eta \in \mathfrak{D}$ and $\eta^*S \cap S \neq \varnothing$ then $\eta \in I_g$.*

Note. The sense in which F is a diffeomorphism is discussed in the footnote on [2, p. 381]. Also the slice of Ebin [3] is essentially $\exp_g$ of the slice (the Palais slice) which we use here where $\exp_g$ is the exponential map of $\mathfrak{M}$ considered as a Riemannian manifold. The Palais slice theorem was not stated in the form of Theorem 4.1 (see [3, Remark, p. 34]), but Ebin's proof goes through if $\exp_g$ is replaced by $h \to g + h$ and yields Theorem 4.1.

Theorem 4.2. *(M, g) is critical $\Leftrightarrow$ (M, g) is semi-solo.*

Proof. We have seen "$\Leftarrow$" (Theorem 2.4). Assume that (M, g) is not semi-solo. Then we know there exists $s \in \Gamma(S^2M)$ such that $s \neq 0$, $\langle s, g\rangle_g = 0$, div $s = 0$, and $\eta^*s = s$ for all $\eta \in I_g$. Define $f\colon S \to \mathbf{R}$ where S is the slice of Theorem 4.1 by $f(g + h) = \langle h, s\rangle_g$ for all $h \in V$, V as in Theorem 4.1. We extend f to a tubular neighborhood of the orbit O_g by means of the formula $f(\eta^*(g + h)) = f(g + h) = \langle h, s\rangle_g$. Now f is well-defined: note $\eta_1^*(g + h_1) = \eta_2^*(g + h_2) \Rightarrow (\eta_1\eta_2^{-1})^*(g + h_1) = g + h_2 \Rightarrow (\eta_1\eta_2^{-1})(S) \cap S \neq \varnothing \Rightarrow \eta_1\eta_2^{-1} \in I_g$. It then follows that $(\eta_1\eta_2^{-1})^*h_1 = h_2$, whence $\langle h_2, s\rangle_g = \langle(\eta_1\eta_2^{-1})^*h_1, s\rangle_g = \langle h_1, (\eta_2\eta_1^{-1})^*s\rangle_g = \langle h_1, s\rangle_g$. Thus f is well-defined. It is clear that f is smooth in every sense. Indeed, it is smooth on S, and in terms of the local chart $F\colon U \times S \to \mathfrak{M}$ of Theorem 4.1, we have $f(F(u, s)) = f(\chi(u)^*s) = f(s)$. It is trivial to check that f is invariant, and, although f is not fully-invariant, we have

$$\begin{aligned} f(\lambda\eta^*(g + h)) &= f(\eta^*(\lambda(g + h))) = f(\eta^*(g + (\lambda - 1)g + \lambda h)) \\ &= \langle(\lambda - 1)g + \lambda h, s\rangle_g = \lambda\langle h, s\rangle_g = \lambda f(\eta^*(g + h)), \end{aligned}$$

where the arguments are in the domain of f. Thus we may extend f to $\mathbf{R}^+\mathfrak{D}^*(S)(= \mathfrak{D}^*(\mathbf{R}^+S))$ via $f(\lambda k) = \lambda f(k)$ for $\lambda > 0$ and $k \in \mathfrak{D}^*(S)$. We can define a fully-invariant function f_0 on $\mathfrak{D}^*(\mathbf{R}^+S)$ by dividing f by another invariant function $k \to \langle k, k\rangle_k^{2/n}$ which is homogeneous of degree 1, obtaining $f_0(k) = f(k)\langle k, k\rangle_k^{-2/n}$ (note $\langle k, k\rangle_k = n\,\mathrm{vol}(M, k)$). We can extend f_0 (retaining its values on a neighborhood of $\mathbf{R}^+O_g$) to a fully-invariant function defined on all of $\mathfrak{M}$ as follows. For $s \in \mathbf{R}^+S$, define $\Phi(s) = \| g\langle g, g\rangle_g^{-2/n} - s\langle s, s\rangle_s^{-2/n}\|_g^p$ where $\| \ \|_g^p$ is the Sobolev norm discussed on [3, p. 21]. If $p > n/2 + 2$, then there is an $\varepsilon > 0$ such that for $h \in H$ (H as in Theorem 4.1), $\Phi(h) < \varepsilon$ implies $h \in \mathbf{R}^+S$. (Recall the definition of S in line one of the proof of [3, Theorems 7.4 and 7.1]. Remember to use $h \to g + h$ in place of $h \to \exp_g h$ in all constructions.) Extend Φ to $\mathfrak{D}^*(\mathbf{R}^+S)$ via $\Phi(\eta^*s) = \Phi(s)$, and note that Φ is well-defined since $\Phi(\zeta^*s) = \Phi(s)$ for $\zeta \in I_g$. Let $\rho\colon \mathbf{R} \to \mathbf{R}$ be C^∞, identically 1 in a neighborhood of O, even, and supported in $(-\varepsilon, \varepsilon)$. Then we may extend the fully-invariant function $(\rho \circ \Phi)f_0$ by zero values from $\mathfrak{D}^*(\mathbf{R}^+S)$ to all of $\mathfrak{M}$. Finally, we must prove $(\nabla f_0)_g \neq 0$, whence (M, g) is not critical, and we are done. Note $(\nabla f_0)_g = \langle g, g\rangle_g^{-n/2}(\nabla f)_g - \frac{n}{2}\langle g, g\rangle^{-n/2-1}\frac{n}{2}gf(g)$. Now $f(g) = \langle 0, s\rangle_g = 0$ while $(\nabla f)_g$ is computed by noting $(\nabla f)_g \in H$ by Lemma 2.5, and for $h \in H$, we have

$$\begin{aligned} \langle(\nabla f)_g, h\rangle_g &= \lim_{t\to 0}\frac{1}{t}(f(g + th) - f(g)) \\ &= \lim_{t\to 0}\frac{1}{t}(\langle th, s\rangle - \langle 0, s\rangle) = \langle h, s\rangle, \end{aligned}$$

whence $(\nabla f_0)_g = \langle g, g\rangle_g^{-n/2}s \neq 0$ as required.

Theorem 4.3. *If (M, g) is semi-solo, then (M, g) is homogeneous.*

Proof. Suppose (M, g) is not homogeneous. Let W be the vector space of all smooth functions on M which are invariant under the isometry group I_g. Now W is certainly infinite dimensional. Indeed consider all functions with support in a sufficiently small tubular neighborhood of an orbit of I_g, and which are smooth functions of the square of the distance to the orbit. All such functions are in W. Now, for all f in a sufficiently small neighborhood of 0 in W, $(1 + f)g$ will be in the image of $F(U \times S)$ (see Theorem 4.1). Thus $(1 + f)g = \chi(u)^*s$ for some $u \in U \subset O_g$ and $s \in S$. Let $\eta = \chi(u)^{-1}$. Then $\eta^*((1 + f)g) = s$. Let I_s be the isometry group of s. Observe that $I_s \subset I_g$ by (2) of the slice theorem. Moreover, the invariance and smallness of f imply that the isometry group of $(1 + f)g$ is the same as that of g. As η: $(M, s) \to (M, (1 + f)g)$ is an isometry, we have that $\Phi \to \eta \circ \Phi \circ \eta^{-1}$ is an isomorphism of I_s onto I_g. This fact, together with $I_s \subset I_g$, yields $I_s = I_g$. Now $s \in S$ implies $s = g + h$ for some h with $\operatorname{div} h = 0$. Since $I_s = I_g$, h is invariant under I_g. We have yet to establish that f can be chosen so that $s = \eta^*((1 + f)g) \neq \lambda g$. If $\eta^*((1 + f)g) = \lambda g$, then η is a conformal transformation of (M, g). The group of such transformations is a finite dimensional Lie group. Hence the family of functions f such that $\eta^*((1 + f)g) = \lambda g$ for some $\eta \in \mathfrak{D}$ depends smoothly on finitely many parameters, and hence cannot contain any neighborhood of 0 in W (by the Baire category theorem). Thus a suitable $f \in W$ can be chosen so that $s \equiv \eta^*((1 + f)g) \neq \lambda g$, and h (defined by $s = g + h$) has all the properties required in order to deduce that (M, g) is not semi-solo.

Theorem 4.4. *If (M, g) is semi-solo, then (M, g) is an* IHS.

Proof. By Theorem 4.3 we already know that (M, g) is homogeneous. Suppose (M, g) is not isotropy-irreducible. Then we have the I_g-invariant $s \in \Gamma(S^2M)$ given by $s(X, Y) = g(\pi X, \pi Y)$ as in the proof of Lemma 2.1; certainly $s \neq \lambda g$. If $\operatorname{div} s = 0$, then (M, g) is not semi-solo, and we are done. If $\operatorname{div} s \neq 0$, then $\Phi \equiv \operatorname{div} s$ is an I_g-invariant one-form. Such a form must be zero if (M, g) is symmetric, but we must do more work if (M, g) is not symmetric. Now $\lambda g \neq \Phi \otimes \Phi \in S^2(M)$, $\Phi \otimes \Phi$ is I_g-invariant, and $\operatorname{div}(\Phi \otimes \Phi) = (\delta \Phi)\Phi + \nabla_x \Phi$, where X is determined by $g(X, \cdot) = \Phi(\cdot)$, and δ denotes codifferential. Since $\delta\Phi = \text{const.}$ and $\int_M \delta\Phi\omega_g = 0$, we have $\delta\Phi = 0$, and so $\operatorname{div}(\Phi \otimes \Phi) = \nabla_x\Phi$. If $\nabla_x\Phi = 0$, we are done. If $\nabla_x\Phi \neq 0$, then $\nabla_x\Phi$ is also an I_g-invariant one-form on M. Moreover, $0 = X\langle\Phi, \Phi\rangle = 2\langle\nabla_x\Phi, \Phi\rangle$ implies that $\nabla_x\Phi \neq \lambda\Phi$. Let Λ be the vector space of I_g-invariant one-forms on M. We have shown that if $\dim \Lambda \leqslant 1$, then we are done. Note that the $\dim \Lambda \equiv k \leqslant n$, since an element of Λ is determined by its value at a point. Consider the symmetric tensor product $\Lambda \circ \Lambda \subset \Gamma(S^2M)$. Certainly elements of $\Lambda \circ \Lambda$

are invariant. Thus div: $\Lambda \circ \Lambda \to \Lambda$ is a well-defined linear map. Moreover,

$$\dim(\ker(\operatorname{div}|\Lambda \circ \Lambda)) \geqslant \dim(\Lambda \circ \Lambda) - \dim \Lambda = \frac{1}{2}k(k+1) - k \geqslant 1,$$

since $k \geqslant 2$. Thus we have some $h \in \ker(\operatorname{div}|\Lambda \circ \Lambda)$ with $h \neq 0$. Certainly $h \neq \lambda g$ unless $k = n$. For $n = 2$, we know that if (M, g) is homogeneous, then (M, g) is symmetric, and recall that we are done with the case where (M, g) is symmetric. If $k = n \geqslant 3$, then $\frac{1}{2}k(k+1) - k \geqslant 3$, and thus in this remaining case we may select $h \in \ker(\operatorname{div}|\Lambda \circ \Lambda)$ such that $h \neq \lambda g$. Thus (M, g) is not semi-solo. q.e.d.

Finally observe that the main Theorem 1.1 follows immediately from Theorems 4.2, 4.4, and 2.3. Indeed we have shown that all of the following properties are the same: solo, semi-solo, critical, and isotropy-irreducible homogeneous.

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